UDC 531.36+62-50

PRINCIPLES OF MINIMUM IN PROBLEMS OF OPTIMAL CONTROL OF RANDOM PROCESSES

PMM Vol. 42, № 2, 1978, pp. 233-244

S.F.MOROZOV and I.P.SMIRNOV
(Gor'kii)

(Received February 8, 1977)

The problem of optimal control by solution of a system of Ito's stochastic integral equations of the general form is considered in the case when control parameters appear in nonhomogeneous terms and coefficients of equations. Necessary conditions of optimality are established (in the form of principles of minimum) for the problem of optimal control with constraints, and an example is presented.

Two trends can be discerned in the theory of control of random processes defined by Ito's stochastic integral equations. The first, developed in [1,2], is related to the derivation of Bellman's equation for the gain function. The second, whose origin can be traced to [3,4], is based on the derivation of necessary conditions of optimality in the form of principles of minimum (stochastic principles of minimum).

The present investigation follows the second trend, and is devoted to the development of a general functional method for stochastic control systems [5,6].

1. Statement of the problem. The control of a process defined by a system of stochastic integral equations of the general form

$$\eta_i(t) = \varphi_i(t, c(t)) + \int_0^t A_i(\tau, \eta(\tau), a(\tau))\, d\tau + \int_0^t B_{ij}(\tau, \eta(\tau), b(\tau))\, dw_j(\tau) \quad (1.1)$$

$$i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m$$

is considered in the case when control functions $a(\tau, \omega)$, $b(\tau, \omega)$ and $c(\tau, \omega)$ appear in coefficients and nonhomogeneous terms of equations. In (1.1) $\varphi_i(t, c, \omega)$, $A_i(t, x, a, \omega)$ and $B_{ij}(t, x, b, \omega)$ denote random fields (ω is the event), the second integrals are understood in Ito's meaning [7], and recurrent subscripts here and below indicate summation.

A particular example of such problem in which $\varphi_i(t, c, \omega) \equiv \varphi_i(\omega)$ and functions $B_{ij}(t, x, b, \omega) \equiv B_{ij}(t)$ (or depend on nonrandom components of solution $\eta(t, \omega)$) was investigated in [3] by a method different from the one used here.

Let (Ω, F, P) be the total probability space and $w(t, \omega)$ be an m-dimensional Wiener process consistent with a nondecreasing set of σ-algebras $F_t \subset F$, $t \geqslant 0$ complete in measure P. We denote by $L_p(\mathrm{B}F)$, $1 \leqslant p < \infty$ the Banach space of the progressively measurable ($\mathrm{B}F$-measurable) over the stream $\{F_t\}$ of random functions $\psi(t, \omega)$, $t \in [0, T]$ and $\omega \in \Omega$ with the finite norm

$$\|\psi\|_p = \Big[\int_{[0, T]\times\Omega} |\psi(t, \omega)|^p\, d(\mathrm{mes} \times P) \Big]^{1/p}$$

Let $H_p(\mathrm{B}F)$ be a set of functions from $L_p(\mathrm{B}F)$ for which $\sup M|\psi(t, \omega)|^p < \infty$, $0 \leqslant t \leqslant T$. Set $H_p(\mathrm{B}F)$ is dense everywhere in $L_p(\mathrm{B}F)$.

Let $L_{p,n}(\mathrm{B}F) = L_p(\mathrm{B}F) \times \ldots \times L_p(\mathrm{B}F)$ (n times) be the space of vector functions $\xi(t, \omega) \equiv \{\xi_1(t, \omega), \ldots, \xi_n(t, \omega)\}$ with norm $\| \xi \|_{p,n}^p = \| \xi_1 \|_p^p + \ldots + \| \xi_n \|_p^p$. Similarly $H_{p,n}(\mathrm{B}F) = H_p(\mathrm{B}F) \times \ldots \times H_p(\mathrm{B}F)$ (n times).

Functions $\varphi_i(t, c, \omega)$, $A_i(t, x, a, \omega)$, and $B_{ij}(t, x, b, \omega)$ have the following properties: 1) they are measurable over the totality of variables when

$$t \in [0, T], \quad x \equiv \{x_1, \ldots, x_n\} \in E^n, \quad \omega \in \Omega, \quad a \equiv \{a_1, \ldots a_{m_1}\} \in D^a \subset E^{m_1}\, b \equiv \{b_1, \ldots, b_{m_2}\} \in D^b \subset E^{m_2}$$
$$c \equiv \{c_1, \ldots, c_{m_3}\} \in D^c \subset E^{m_3}$$

where D^a, D^b, and D^c are bounded sets in respective Euclidean spaces; 2) they are F_t-measurable over ω for fixed t, x, a, b, c from $[0, T] \times E^n \times D^a \times D^b \times D^c$; and 3) for fixed t, x, and ω they are continuous over the totality of remaining variables. Let us assume that functions $A_i(t, x, a, \omega)$ and $B_{ij}(t, x, b, \omega)$ are differentiable with respect to variables x_k and that their derivatives A_{x_k}' and B_{x_k}' are continuous over the totality of variables (x, a) and (x, b) respectively, and their absolute values are uniformly bounded by the constant number $N > 0$ for all $t, x, a, b,$ and ω in the determination region. We also assume that $| \varphi_i(t, c, \omega) | \leqslant \alpha_i(t, \omega) \in H_r(\mathrm{B}F)$, $r > 2$ for all $c \in D^c$.

For every $t \in [0, T]$ we specify sets $D^a(t)$, $D^b(t)$, and $D^c(t)$ of random vectors that are F_t-measurable and assume the values D^a, D^b, and D^c respectively. We assume that sets $D^i(t)$, $i = a, b, c$, are nondecreasing and continuous on the left with respect to $t : D^i(t) = D^i(0) \bigcup D^i(s)$, $0 < s < t$.

We consider vector functions $a(t, \omega)$, $b(t, \omega)$, and $c(t, \omega)$ to be admissible controls, if their components are $\mathrm{B}F$-measurable and

$$a(t, \omega) \in D^a(t), \; b(t, \omega) \in D^b(t), \; c(t, \omega) \in D^c(t), \; \forall t$$

A random process $H_{s,n}(\mathrm{B}F)$ that satisfies almost certainly (1.1) for every $t \in [0, T]$ is called the solution of system (1.1) which corresponds to the admissible control $u(t, \omega) \equiv \{a(t, \omega), b(t, \omega), c(t, \omega)\}$ (see [7]).

Let us consider the optimization problem of finding among the admissible controls $u(t, \omega)$ that satisfy the constraint

$$L(u) = M \int_0^T L(s, \eta_u(s), u(s))\, ds \leqslant 0$$

a control that minimizes the functional

$$J(u) = M \int_0^T J(s, \eta_u(s), u(s))\, ds$$

where $\eta_u(s)$ is a solution of system (1.1) which corresponds to control $u(s)$ and M is the symbol of mathematical expectation.

Let functions $L(t, x, a, b, c, \omega)$ and $J(t, x, a, b, c, \omega)$ satisfy the following conditions: 1) be measurable over the totality of variables for $t, x, a, b, c, \omega \in [0,T] \times E^n \times D^a \times D^b \times D^c \times \Omega$; 2) be F_t-measurable with respect to ω for fixed t, x, a, b, c; and 3) be continuous over the totality of variables

(a, b, c) for almost all t, x, and ω from $[0, T] \times E^n \times \Omega$ and be twice differentiable with respect to variables x_k. Let furthermore the following conditions for the order of increase with respect to x:

$$|J|, |L| \leqslant K_1(1+|x|^\rho), \quad |J^{\bullet}_{x_k}|, |L^{\bullet}_{x_k}| \leqslant K_2(1+|x|^{\rho-1})$$

$$|J''_{x_k x_j}|, |L^{\bullet}_{x_k x_j}| \leqslant K_3(1+|x|^{\rho-2}), \quad \rho \leqslant (r+2)/2$$

where K_1, K_2, and K_3 are constant numbers, be satisfied for almost all t, a, b, c, and ω from the determination region.

2. Variation of controls. Let $u^\circ(t, \omega) \equiv \{a^\circ(t, \omega), b^\circ(t, \omega), c^\circ(t, \omega)\}$ be the optimal admissible control and $\eta^0(t, \omega)$ the relates solution of system (1.1). Let us derive a variant of control $u^\varepsilon(t, \omega) \equiv \{a^\varepsilon(t, \omega), b^\varepsilon(t, \omega), c^\varepsilon(t, \omega)\}$. Let $\tau_0 \in [0, T]$. We take a finite set of points $\{\tau_k\}$ from the interval (τ_0, T). Let $\{\alpha_k^l\}$ be an arbitrary finite set of nonnegative numbers and $\{a_k^l(\omega)\}$, $\{b_k^l(\omega)\}$, and $\{c_k^l(\omega)\}$ be sets of random vectors from $D^a(\tau_0)$, $D^b(\tau_0)$, and $D^c(\tau_0)$ respectively. We define the component $a^\varepsilon(t, \omega)$ of the control variant by

$$a^\varepsilon(t, \omega) = \begin{cases} a_k^l(\omega), & t \in \Pi_k^l(\tau_k) \\ a^\circ(t, \omega), & t \in [0, T] \setminus \bigcup\limits_{l, k} \Pi_k^l(\tau_k) \end{cases}$$

$$\Pi_k^l(\tau_k) = \left[\tau_k - \varepsilon \sum_{i=1}^{l} \alpha_k^i, \quad \tau_k - \varepsilon \sum_{i=1}^{l-1} \alpha_k^i\right)$$

For fairly small ε, $0 < \varepsilon \leqslant \varepsilon_\alpha$, intervals Π_k^l evidently lie in (τ_0, T) and do not intersect pairwise. Components $b^\varepsilon(t, \omega)$ and $c^\varepsilon(t, \omega)$ of variant $u^\varepsilon(t, \omega)$ with corresponding parameters $\{b_k^l(\omega)\}$ and $\{c_k^l(\omega)\}$ are similarly defined.

The following theorem on the existence of solution of system (1.1) and its stability with respect to control perturbations is valid.

Theorem 1. If functions $\varphi_i(t, c, \omega)$, $A_i(t, x, a, \omega)$ and $B_{ij}(t, x, b, \omega)$ satisfy the conditions formulated above, then there exists a positive number ε^*, $0 < \varepsilon^* \leqslant \varepsilon_\alpha$ such that to each variant $u^\varepsilon(t, \omega)$ corresponds for $0 \leqslant \varepsilon \leqslant \varepsilon^*$ the unique solution $\eta^\varepsilon(t, \omega) \in H_{r,n}(\mathrm{B}F)$ of system (1.1) and $\|\eta^\varepsilon - \eta^\circ\|_{r,n} \to 0$ when $\varepsilon \to 0$.

Proof. The existence an uniqueness of solution of system (1.1) directly follows from the theorem in [7]. Since for any admissible control function $\varphi(t, c(t)) \in H_{r,n}(\mathrm{B}F)$ hence also the related solution $\eta(t) \in H_{r,n}(\mathrm{B}F)$.

To prove the stability of the optimal solution we consider the system of linear integral stochastic equations

$$\xi_i(t) = \psi_i(t) + \int_0^t \frac{\partial A_i}{\partial x_k}(s, \eta_{1ik}^\varepsilon(s), a^\varepsilon(s))\, \xi_k(s)\, ds + \tag{2.1}$$

$$\int_0^t \frac{\partial B_{ij}}{\partial x_k}(s, \eta_{2ijk}^{\varepsilon}(s), b^{\varepsilon}(s))\, \xi_k(s)\, dw_j(s)$$

where the "average" points $\eta_{1ik}{}^{\varepsilon}$ and $\eta_{2ijk}{}^{\varepsilon}$ are determined with unit probability by the equalities

$$A_i(t, \eta^{\varepsilon}(t), a^{\varepsilon}(t)) - A_i(t, \eta^{\circ}(t), a^{\varepsilon}(t)) = \frac{\partial A_i}{\partial x_k}(t, \eta_{1ik}^{\varepsilon}(t), a^{\varepsilon}(t))\, \Delta_{\varepsilon}\eta_k(t) \tag{2.2}$$

$$B_{ij}(t, \eta^{\varepsilon}(t), b^{\varepsilon}(t)) - B_{ij}(t, \eta^{\circ}(t), b^{\varepsilon}(t)) = \frac{\partial B_{ij}}{\partial x_k}(t, \eta_{2ijk}^{\varepsilon}(t), b^{\varepsilon}(t))\, \Delta_{\varepsilon}\eta_k(t)$$

$$(\Delta_{\varepsilon}\eta_k(t) \equiv \eta_k{}^{\varepsilon}(t) - \eta_k{}^{\circ}(t))$$

Lemma 1. The solution $\xi(t) \in H_{p,n}(\mathrm{B}F)$ of system (2.1) exists and is unique for any function $\psi(t) \in H_{p,n}(\mathrm{B}F)$, $p \geqslant 2$, and the estimate

$$\| \xi \|_{p,n} \leqslant C_p \| \psi \|_{p,n}, \quad C_p = C_p(p, m, n, T, N) \tag{2.3}$$

is valid.

Proof. The existence and uniqueness of solution of system (2.1) follows from the same theorem in [7]. By applying the Hölder inequality to indentity (2.1) we obtain

$$|\xi_i(t)|^p \leqslant 3^{p-1}\left\{ |\psi_i(t)|^p + \left| \int_0^t \frac{\partial A_i}{\partial x_k}(\cdot)\, \xi_k(s)\, ds \right|^p + \left| \int_0^t \frac{\partial B_{ij}}{\partial x_k}(\cdot)\, \xi_k(s)\, dw_j(s) \right|^p \right\}$$

from which, using the estimate of Ito's integral of moments [8], we have

$$M|\xi_i(t)|^p \leqslant 3^{p-1}\left\{ M|\psi_i(t)|^p + C_p' \int_0^t \sum_{k=1}^n M|\xi_k(s)|^p ds \right\}$$

Summing by i, applying Gronwall's lemma and integrating along $[0, T]$, we obtain the required estimate (2.3).

To prove the theorem it remains to note that the random vector function $\xi(t) = \eta^{\varepsilon}(t) - \eta^{\circ}(t)$ satisfies system (2.1) when

$$\psi_i(t) = \Delta_u \Phi_i(t) \equiv \Delta_c \varphi_i(t) + \int_0^t \Delta_a A_i(s)\, ds + \int_0^t \Delta_b B_{ij}(s)\, dw_j(s) \in H_{r,n}(\mathrm{B}F)$$

$$\Delta_c \varphi_i(t) \equiv \varphi_i(t, c^{\varepsilon}(t)) - \varphi_i(t, c^{\circ}(t))$$

$$\Delta_a A_i(t) \equiv A_i(t, \eta^{\circ}(t), a^{\varepsilon}(t)) - A_i(t, \eta^{\circ}(t), a^{\circ}(t))$$

$$\Delta_b B_{ij}(t) \equiv B_{ij}(t, \eta^{\circ}(t), b^{\varepsilon}(t)) - B_{ij}(t, \eta^{\circ}(t), b^{\circ}(t))$$

Then on the basis of (2.3) we have

$$\| \eta^\varepsilon - \eta^\circ \|_{r,n} \leqslant C_r \| \Delta_u \Phi \|_{r,n} \to 0, \quad \varepsilon \to 0$$

3. **Derivation of integral representations $\Delta_\varepsilon J$ and $\Delta_\varepsilon L$.** By Lemma 1 the linear system (2.1) determines the linear operator $R_\varepsilon' : H_{p,n}(\mathrm{B}F) \to H_{p,n}(\mathrm{B}F)$, $\xi = R_\varepsilon' \psi$. Since the set $H_{p,n}(\mathrm{B}F)$ is dense everywhere in $L_{p,n}(\mathrm{B}F)$ and estimate (2.3) is valid, hence R_ε' permits the extension in continuity of R_ε in $L_{p,n}(BF)$, i.e. the continous operator $R_\varepsilon : L_{p,n}(\mathrm{B}F) \to L_{p,n}(\mathrm{B}F)$, $p \geqslant 2$, $\| R_\varepsilon \| \leqslant C_p$ has been determined.

We define the increment $\Delta_\varepsilon \eta(t)$ as follows:

$$\Delta_\varepsilon \eta = R_\varepsilon \Delta_u \Phi \tag{3.1}$$

To determine the first variations of functionals J and L we define the integral representations of increments by

$$\Delta_\varepsilon J(u) \equiv J(u^\varepsilon) - J(u^\circ), \ \Delta_\varepsilon L(u) \equiv L(u^\varepsilon) - L(u^\circ)$$

As an example let us consider $\Delta_\varepsilon J$

$$\Delta_\varepsilon J = M \int_0^T \frac{\partial J}{\partial x_k}(s, \eta_{3k}^\varepsilon(s), u^\varepsilon(s)) \Delta_\varepsilon \eta_k(s)\, ds + M \int_0^T \Delta_u J(s)\, ds \equiv \tag{3.2}$$

$$\Delta_\varepsilon^{(1)} J + \Delta_\varepsilon^{(2)} J$$

$$(\Delta_u J(t) \equiv J(t, \eta^0(t), u^\varepsilon(t)) - J(t, \eta^\circ(t), u^\circ(t)))$$

where the average points $\eta_{3k}{}^\varepsilon$ are determined by equalities of the form (2.2) for function J.

We determine in $L_{2,n}(\mathrm{B}F)$ the set (in ε) of bounded linear functionals

$$\{T_\varepsilon\}: (T_\varepsilon, \psi) = M \int_0^T \frac{\partial J}{\partial x_k}(s, \eta_{3k}^\varepsilon(s), u^\varepsilon(s)) \psi_k(s)\, ds$$

Using the Riesz theorem on the linear functional representation, by virtue of (3.1) we obtain

$$\Delta_\varepsilon^{(1)} J = (T_\varepsilon, \Delta_\varepsilon \eta) = (T_\varepsilon, R_\varepsilon \Delta_u \Phi) = (R_\varepsilon{}^* T_\varepsilon, \Delta_u \Phi) \equiv (Q_\varepsilon, \Delta_u \Phi) =$$

$$M \int_0^T \chi_i{}^\varepsilon(t) \Delta_u \Phi_i(t)\, dt, \qquad \chi^\varepsilon = R_\varepsilon{}^* T_\varepsilon \in L_{2,n}(\mathrm{B}F)$$

Substituting this expression into (3.2) we obtain the sought integral representation

$$\Delta_\varepsilon J = M \int_0^T \chi_i{}^\varepsilon(t) \Delta_u \Phi_i(t)\, dt + M \int_0^T \Delta_u J(t)\, dt \tag{3.3}$$

Let us show that for $1 < q' < 2$

$$\| \chi^\varepsilon - \chi^0 \|_{q',n} \to 0, \quad \varepsilon \to 0 \tag{3.4}$$

where

$$\chi^\circ = R_0{}^* T_0, \quad (T_0, \psi) = M \int_0^T \frac{\partial J}{\partial x_k} (s, \eta^\circ(s), u^\circ(s)) \psi_k(s)\, ds$$

and $R_0 : L_{p,n}(\mathrm{B}F) \to L_{p,n}(\mathrm{B}F), \quad p \geqslant 2$ is a linear operator linked to system

$$\xi_i(t) = \psi_i(t) + \int_0^t \frac{\partial A_i}{\partial x_k}(s, \eta^\circ(s), a^\circ(s))\, \xi_k(s)\, ds + \int_0^t \frac{\partial B_{ij}}{\partial x_k}(s, \eta^\circ(s), b^\circ(s))\, \xi_k(s)\, dw_j(s) \tag{3.5}$$

We have $(q > 2,\ 1/q + 1/q' = 1)$

$$\begin{aligned} \| \chi^\varepsilon - \chi^\circ \|_{q', n} \leqslant \| R_\varepsilon \|_{L_{q,n} \to L_{2,n}} \| T_\varepsilon - T_0 \|_{(L_{2,n})^*} + \\ \| T_0 \|_{(L_{2,n})^*} \| R_\varepsilon - R_0 \|_{L_{q,n} \to L_{2,n}} \end{aligned} \tag{3.6}$$

We shall show that the last operator norm in (3.6) tends to zero when $\varepsilon \to 0$. We take a monotonically decreasing sequence of positive numbers $\delta_m \to 0$ with $m \to \infty$. For every function φ in a unit sphere in $L_{q,n}(\mathrm{B}F)$ there exists a sequence $\{\varphi_m\} \subset H_{q,n}(\mathrm{B}F)$ such that $\| \varphi - \varphi_m \|_{q,n} \leqslant \delta_m$. Let $\{\xi^{\varepsilon m}\}$, $\{\xi^{0m}\} \subset H_{q,n}(\mathrm{B}F)$ be sequences of solutions (2.1) and (3.5), respectively, that correspond to $\{\varphi_m\}$. Then

$$\begin{aligned} \| (R_\varepsilon - R_0)\varphi \|_{2,n} \leqslant \| R_\varepsilon(\varphi - \varphi_m) \|_{2,n} + \| R_0(\varphi - \varphi_m) \|_{2,n} + \\ \| R_\varepsilon \varphi_m - R_0 \varphi_m \|_{2,n} \leqslant 2C_2 T^{(q-2)/2q} \delta_m + \| \xi^{\varepsilon m} - \xi^{0m} \|_{2,n} \end{aligned} \tag{3.7}$$

Using the estimate of moments of Ito's integral, Gronwall's lemma, and Hölder's inequality, we obtain for the norm the estimate

$$\| \xi^{\varepsilon m} - \xi^{0m} \|_{2,n}{}^2 \leqslant K_4\, C_q^2 (1 + \delta_m)^2 \| \Gamma_\varepsilon \|_{q/(q-2)}.$$

$$\Gamma_\varepsilon(t) \equiv T \sum_{i,k} \left\{ \left| \frac{\partial A_i}{\partial x_k}(t, \eta^\varepsilon_{1ik}(t), a^\varepsilon(t)) - \frac{\partial A_i}{\partial x_k}(t, \eta^\circ(t), a^\circ(t)) \right|^2 + \sum_j B_2 \left| \frac{\partial B_{ij}}{\partial x_k}(t, \eta^\varepsilon_{2ijk}(t), b^\varepsilon(t)) - \frac{\partial B_{ij}}{\partial x_k}(t, \eta^\circ(t), b^\circ(t)) \right|^2 \right\}$$

$(K_4,\ B_2 = \mathrm{const})$

Since $|\eta^\varepsilon_{1ik} - \eta^\circ|$, $|\eta^\varepsilon_{2ijk} - \eta^\circ|$, and $|u^\varepsilon - u^\circ| \to 0$, $\varepsilon \to 0$ in proportion to $\mathrm{mes} \times P$ (Theorem 1), functions A'_{x_k} and B'_{x_k} are continuous with respect to (x, a) and (x, b), respectively, and are bounded, hence $\| \Gamma_\varepsilon \| \to 0$ when $\varepsilon \to 0$.

It now follows from (3.7) that $\| (R_\varepsilon - R_0)\varphi \|_{2,n} \to 0$ uniformly when $\varepsilon \to 0$ for φ selected from the unit sphere, which means that

$$\| R_\varepsilon - R_0 \|_{L_{q,n} \to L_{2,n}} \to 0, \quad \varepsilon \to 0 \tag{3.8}$$

The conditions imposed on the integrand of functional J ensures the following passing to limit:

$$\| T_\varepsilon - R_0 \|_{(L_{2,n})^*} \to 0, \quad \varepsilon \to 0 \tag{3.9}$$

Thus (3.4) follows from (3.9), (3.8), and (3.6).

4. First variations of functionals. We define variations of functionals as the limits

$$\delta J = \lim_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k} \Delta_{\varepsilon_k} J, \quad \delta L = \lim_{\delta_k \to 0} \frac{1}{\delta_k} \Delta_{\delta_k} L$$

where $\{\varepsilon_k\}$ and $\{\delta_k\}$ are some subsequencies of the sequence $\varepsilon \to 0$. From (3.3) we have

$$\Delta_\varepsilon^{(2)} J = M \sum_{l,k} \int_{\Pi_k^l(\tau_k)} [J(t, \eta^\circ(t), a_k^l, b_k^l, c_k^l) - J(t, \eta^\circ(t), u^\circ(t))]\, dt$$

$$\Delta_\varepsilon^{(1)} J = M \sum_{l,k} \int_{\Pi_k^l(\tau_k)} [\chi_i^\varepsilon(t) \Delta_c \varphi_i(t) + \theta_i^\varepsilon(t) \Delta_a A_i(t)]\, dt +$$

$$M \int_0^T \chi_i^\varepsilon(t) \Big[\int_0^t \Delta_b B_{ij}(s)\, dw_j(s)\Big]\, dt \equiv \Delta_\varepsilon^{(3)} J + \Delta_\varepsilon^{(4)} J$$

$$\Big(\theta_i^\varepsilon(t) \equiv \int_t^T \chi_i^\varepsilon(s)\, ds\Big)$$

Using (3.4) it is possible to show that for any pair of integers N and M and an arbitrary set of random quantities $\{a_k^l(\omega)\}$, $\{b_k^l(\omega)\}$, and $\{c_k^l(\omega)\}$, $k = 1, 2, \ldots, N$ and $l = 1, 2, \ldots, M$ there exists a set $E_u^{NM} \subset (\tau_0, T)$, $\operatorname{mes} E_u^{NM} = T - \tau_0$ such that for any arbitrary set of points $\{\tau_k\} \subset E_u^{NM}$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\Delta_\varepsilon^{(2)} J + \Delta_\varepsilon^{(3)} J) = \sum_{l,k} \alpha_k^l [g_1^{lk}(\tau_k) + g_2^{lk}(\tau_k)] \tag{4.1}$$

$$g_1^{lk}(\tau) = M[J(\tau, \eta^\circ(\tau), a_k^l, b_k^l, c_k^l) - J(\tau, \eta^\circ(\tau), u^\circ(\tau))]$$

$$g_2^{lk}(\tau) = M\{\chi_i^\circ(\tau)[\varphi_i(\tau, c_k^l) - \varphi_i(\tau, c^\circ(\tau))] + \theta_i^\circ(\tau) \times$$
$$[A_i(\tau, \eta^\circ(\tau), a_k^l) - A_i(\tau, \eta^\circ(\tau), a^\circ(\tau))]\}$$

For passing to limit in $(1/\varepsilon)\, \Delta_\varepsilon^{(4)} J$ we impose additional constraints on the stream $\{F_t\}$ of σ-algebras and make the necessary assumptions, namely that

$$F_t \equiv \overline{\sigma[w(s), s \leqslant t]} \tag{4.2}$$

Lemma 2. If condition (4.2) is satisfied and $\varphi(t, \omega) \in L_2(\mathrm{B}F)$, there exists a unique function $\lambda(t, s, \omega) \in L_{2,m}(\mathrm{B} \times \mathrm{B}F)$ such that mes $\times P$ -nearly everywhere in $[0, T] \times \Omega$

$$\varphi(t, \omega) = M\varphi(t, \omega) + \int_0^t \lambda_j(t, s, \omega)\, dw_j(s, \omega) \tag{4.3}$$

Operator $G: \lambda = G\varphi$ from $L_2(\mathrm{B}F)$ is then bounded in $L_{2,m}(\mathrm{B} \times \mathrm{B}F)$ and $\|G\| = 1$. Here $L_{2,m}(\mathrm{B} \times \mathrm{B}F)$ is the L_2-space of m-measurable random fields with $\mathrm{B} \times \mathrm{B}F$-measurable (B is the Borel σ-algebra in E^1) components.

Proof. Let $\eta(t, \omega) = \chi_{[0,t_1)}(t)\eta_0(\omega) + \ldots + \chi_{[t_{n-1},T]}(t)\eta_{n-1}(\omega) \in L_2(BF)$, where $\{t_k\}$, $k = 0, 1, \ldots, n$ represents the finite subdivision of segment $[0, T]$,

$\chi_{[\alpha,\beta)}(t)$ is the indicator function of the half-interval $[\alpha, \beta)$, and $\eta_k(\omega) - F_{t_k}$ is a measurable random quantity (with finite second moment). If conditions (4.2) are satisfied, then according to [9], there exists for each t_k a $\lambda^{(k)}(s, \omega) \in L_{2,m}([0, t_k] \times \Omega, BF)$ such that almost certainly

$$\eta_k(\omega) = M\eta_k(\omega) + \int_0^{t_k} \lambda_j^{(k)}(s, \omega)\, dw_j(s, \omega)$$

Continuing $\lambda^{(k)}(s, \omega)$ in $[0, T] \times \Omega$ through zero and setting $\mu(t, s, \omega) = \chi_{[0,t_1)}(t)\lambda^{(1)}(s, \omega) + \ldots + \chi_{[t_{n-1},T]}(t)\lambda^{n-1}(s, \omega)$ we establish the lemma for step functions. Let $\varphi(t, \omega)$ be an arbitrary function from $L_2(BF)$ which we approximate by a sequence of step functions $\| \eta_n - \varphi \|_2 \to 0$ (see [10]) when $n \to \infty$. Let $\mu^{(n)}(t, s, \omega)$ be a sequence of stochastic kernels that corresponds to $\{\eta_n\}$. From (4.3) we then have

$$\int_0^t [\mu_j^{(n)}(t, s, \omega) - \mu_j^{(k)}(t, s, \omega)]\, dw_j(s, \omega) =$$
$$\eta_n(t, \omega) - \eta_k(t, \omega) + M[\eta_k(t, \omega) - \eta_n(t, \omega)]$$

from which, using the properties of Ito's integral, we obtain

$$\| \mu^{(n)} - \mu^{(k)} \|_{L_{2,m}(B\times BF)} \leqslant 2 \| \eta_n - \eta_k \|_2 \to 0, \quad k, n \to \infty$$

The sequence $\{\mu^{(n)}\}$ is thus fundamental in $L_{2,m}(B \times BF)$ and, because of completeness of that space, it converges to some element λ of the latter. The Fubini theorem on stochastic integrals (see [11]) shows that the stochastic integral

$$\int_0^t \lambda_j(t, s, \omega)\, dw_j(s, \omega)$$

is determinate and BF-measurable with respect to t and ω. Finally

$$\| \varphi - M\varphi - \int_0^t \lambda_j(t, s, \omega)\, dw_j(s, \omega) \|_2 \leqslant \| \varphi - \eta_n \|_2 +$$
$$\| M(\varphi - \eta_n) \|_{L_2(0,T)} + \| \lambda - \mu^{(n)} \|_{L_{2,m}(B\times BF)} \to 0, \quad n \to \infty$$

By raising the identity (4.3) to the second power and integrating with respect to measure P, we obtain a confirmation of the second statement of the lemma.

Applying the proved lemma to the set ($\varepsilon > 0$) of functions $\chi^\varepsilon(t, \omega) \in L_{2,n}(BF)$, we obtain a set of matrices $\{\lambda_{ij}^\varepsilon(t, s, \omega)\} : \{\lambda_{i1}^\varepsilon, \ldots, \lambda_{im}^\varepsilon\} = G\chi_i^\varepsilon$. We introduce the notation

$$\varkappa_{ij}^\varepsilon(t, \omega) \equiv \int_t^T \lambda_{ij}^\varepsilon(s, t, \omega)\, ds \in L_2(BF)$$

and shall show that for $1 < \delta < 2$

$$\| \varkappa_{ij}^\varepsilon - \varkappa_{ij}^\circ \|_\delta \to 0, \quad \varepsilon \to 0 \tag{4.4}$$

Applying twice the Hölder inequalities and the lower bound estimate for the δ-moment of Ito's integral [8], we obtain the sequence of inequalities

$$\| \varkappa_{ij}^{\varepsilon} - \overset{\circ}{\varkappa}_{ij} \|_{\delta}^{\delta} \equiv M \int_{0}^{T} | \varkappa_{ij}^{\varepsilon} - \overset{\circ}{\varkappa}_{ij} |^{\delta} ds =$$

$$M \int_{0}^{T} \left| \int_{s}^{T} [\lambda_{ij}^{\varepsilon}(t, s, \omega) - \overset{\circ}{\lambda}_{ij}(t, s, \omega)] \, dt \right|^{\delta} ds \leqslant$$

$$M \int_{0}^{T} ds \left\{ \int_{s}^{T} | \lambda^{\varepsilon}(\cdot) - \lambda^{\circ}(\cdot) |^{\delta} dt \, (T - s)^{\delta - 1} \right\} \leqslant$$

$$D_{\delta} M \int_{0}^{T} \left\{ s^{\delta/2 - 1} \int_{s}^{T} | \lambda^{\varepsilon}(\cdot) - \lambda^{\circ}(\cdot) |^{\delta} dt \right\} ds \leqslant$$

$$D_{\delta} M \int_{0}^{T} ds \left\{ \int_{s}^{T} t^{\delta/2 - 1} | \lambda^{\varepsilon}(\cdot) - \lambda^{\circ}(\cdot) |^{\delta} dt \right\} \leqslant$$

$$D_{\delta} M \int_{0}^{T} \left| \int_{0}^{t} | \lambda^{\varepsilon}(\cdot) - \lambda^{\circ} |^{2} ds \right|^{\delta/2} dt \leqslant$$

$$\frac{D_{\delta}}{A_{\delta}} \int_{0}^{T} dt \, M \left| \int_{0}^{t} [\lambda^{\varepsilon}(\cdot) - \lambda^{\circ}(\cdot)] \, dw(s) \right|^{\delta} \leqslant$$

$$\frac{D_{\delta}}{A_{\delta}} 2^{\delta} \| \chi^{\varepsilon} - \chi^{\circ} \|_{\delta, n}^{\delta} \quad (A_{\delta} = A_{\delta}(\delta) > 0)$$

where the inequality

$$(T - t)^{\delta - 1} \leqslant D_{\delta} t^{\delta/2 - 1}, \quad 0 \leqslant t \leqslant T, \quad 0 < D_{\delta} = D_{\delta}(\delta) < \infty$$

is also used. Hence (4.4) follows from (3.4).

Reverting to $\Delta_{\varepsilon}^{(4)} J$ we now have

$$\Delta_{\varepsilon}^{(4)} J = M \int_{0}^{T} \chi_{i}^{\varepsilon}(t) \left[\int_{0}^{t} \Delta_{b} B_{ij}(s) \, dw_{j}(s) \right] dt =$$

$$\int_{0}^{T} dt \, M \int_{0}^{t} \lambda_{ij}^{\varepsilon}(t, s) \, dw_{j}(s) \int_{0}^{t} \Delta_{b} B_{ij}(s) \, dw_{j}(s) =$$

$$\int_{0}^{T} dt \, M \int_{0}^{t} \lambda_{ij}^{\varepsilon}(t, s) \, \Delta_{b} B_{ij}(s) \, ds = M \int_{0}^{T} \Delta_{b} B_{ij}(s) \, \varkappa_{ij}^{\varepsilon}(s) \, ds =$$

$$\sum_{l, k} M \int_{\Pi_{k}^{l}(\tau_{k})} \operatorname{Tr} \Delta_{b} B(s) \, (\varkappa^{\varepsilon}(s))' \, ds$$

Using (4.4) it is possible to show that for any pair of integers N and M and an arbitrary set $\{b_{k}^{l}(\omega)\}$ there exists a set $E_{b}^{NM} \subset [\tau_{0}, T]$ of complete measure $\operatorname{mes} E_{b}^{NM} = T - \tau_{0}$ such that for an arbitrary set of points $\{\tau_{k}\} \subset E_{b}^{NM}$

$$\lim_{\theta_{k} \to 0} \frac{1}{\theta_{k}} \Delta_{\theta_{k}}^{(4)} J = \sum_{l, k} \alpha_{k}^{l} g_{3}^{lk}(\tau_{k})$$

$$g_3^{lk}(\tau) = M\,\mathrm{Tr}\,[B(\tau, \eta^\circ(\tau), b_k^l) - B(\tau, \eta^\circ(\tau), b^\circ(\tau))]\,(\varkappa^\circ(\tau))'$$

where Tr denotes the taking of a matrix spur and the prime denotes a transposition.

Taking into account (4.1) we find that for $\tau_k \in E_u^{NM} \cap E_b^{NM}$ the first variation of functional J may be represented as

$$\delta J = \sum_{l,k} \alpha_k^l\,[g_1^{lk}(\tau_k) + g_2^{lk}(\tau_k) + g_3^{lk}(\tau_k)] \tag{4.5}$$

The determination region in (4.5) depends on numbers N and M and the specific sets $\{a_k^l\}$, $\{b_k^l\}$, and $\{c_k^l\}$. To eliminate that dependence we separate in $L_2(\Omega, F, P)$ the denumerable and everywhere dense net of random quantities S_2. It can be shown that transitions to limit which determine the variation δJ are realized in some subsequence $\{\varepsilon_k\}$ on set $E \subset (\tau_0, T)$, mes $E = T - \tau_0$ for any arbitrary admissible sets

$$\{a_k^l\} \subset D^a(\tau_0) \cap S_{2,m_1}, \quad \{b_k^l\} \subset D^b(\tau_0) \cap S_{2,m_2}$$
$$\{c_k^l\} \subset D^c(\tau_0) \cap S_{2,m_3}$$
$$(S_{2,n} \equiv S_2 \times \ldots \times S_2\ (n \text{ times}))$$

Introducing by analogy to T_ε and χ^ε the pair P_ε, $\nu^\varepsilon = R_\varepsilon^* P_\varepsilon$, we determine variation δL.

5. The principles of minimum. Let us consider in E^2 the set of pairs $Q = (\delta J, \delta L)$ which is obtained for all possible admissible sets of points $\{\tau_k\} \subset E$, numbers $\{\alpha_k^l\}$, and random quantities $\{a_k^l\}$, $\{b_k^l\}$ and $\{c_k^l\}$. It can be shown that Q is a convex cone whose vertex is at the coordinate origin. Let $R = \{x_1, x_2 : x_1 < 0,\ x_2 < 0\} \subset E^2$. Optimality of the admissible control $u^\circ(t, \omega)$ implies that cones Q and R do not intersect, which shows that they are separated by some straight line with $\mu = (\mu_1, \mu_2)$, $\mu_1^2 + \mu_2^2 \neq 0$ as its normal vector. Directing that vector toward cone Q we obtain

$$\mu_1 \delta J + \mu_2 \delta L \geqslant 0 \tag{5.1}$$

Let us specify the particular set of parameters

$$\tau_1 \subset E;\ a_1^1(\omega),\ b_1^1(\omega),\ c_1^1(\omega);\ \alpha_1^1 = 1$$

Calculating the related set of variations with allowance for the continuity of functions A_i, B_{ij}, and φ_i with respect to a, b, and c, respectively, and for the properties of sets $D^i(t)$, $i = a, b, c$, we obtain from (5.1) the following definition of the principles of minimum.

The principles of minimum. If $u^\circ(t, \omega)$ is the optimal admissible control and $\eta^\circ(t, \omega) \in H_{r,n}(\mathrm{B}F)$, $r > 2$ is the related optimal solution of system (1.1), then there exist functions χ° and $\nu^\circ \in L_{2,n}(\mathrm{B}F)$ that are uniquely determined by the operator formulas

$$\chi^\circ = R_0^* T_0, \quad \nu^\circ = R_0^* P_0$$

and nonnegative numbers $\mu_1, \mu_2\ (\mu_1^2 + \mu_2^2 \neq 0)$ such that almost everywhere in $[0, T]$

$$
\begin{aligned}
& MH\,(t,\ \eta^{\circ}(t,\ \omega),\ a^{\circ}(t,\ \omega),\ b^{\circ}(t,\ \omega),\ c^{\circ}(t,\ \omega),\ \omega) = \\
& \quad \inf MH\,(t,\ \eta^{\circ}(t,\ \omega),\ a(\omega),\ b(\omega),\ c(\omega),\ \omega) \qquad (5.2) \\
& H\,(t,\eta^{\circ}(t,\omega),\zeta_1,\zeta_2,\zeta_3,\omega) = \mu_1\,[J\,(t,\eta^{\circ}(t,\omega),\zeta_1,\zeta_2,\zeta_3,\omega) + \\
& \quad \chi_i^{\circ}(t,\omega)\,\varphi_i\,(t,\zeta_3,\omega) + \theta_i^{\circ}(t,\omega)\,A_i\,(t,\eta^{\circ}(t,\omega),\zeta_1,\omega) + \\
& \quad \operatorname{Tr} B\,(t,\eta^{\circ}(t,\omega),\zeta_2,\omega)\,(\varkappa^{\circ}(t,\omega))'] + \mu_2\Big[L\,(t,\eta^{\circ}(t,\omega),\zeta_1,\zeta_2,\zeta_3,\omega) + \\
& \quad \nu_i^{\circ}(t,\omega)\,\varphi_i\,(t,\zeta_3,\omega) + \int_t^T \nu_i^{\circ}(s,\omega)\,ds A_i\,(t,\eta^{\circ}(t,\omega),\zeta_1,\omega) + \\
& \quad \operatorname{Tr} B\,(t,\eta^{\circ}(t,\omega),\zeta_2,\omega)\,(\alpha^{\circ}(t,\omega))'\Big]
\end{aligned}
$$

where $\inf$ is taken with respect to all $a\,(\omega)\in D^a(t)$, $b\,(\omega)\in D^b(t)$, and $c(\omega)\in D^c(t)$.

Notes. 1°. It is possible to obtain from (5.2) incomplete principles of minimum for any two groups of control parameters, as well as for each group separately. For instance, to obtain the principles of minimum for the group of parameters $\{a,\ c\}$ it is sufficient to set in the right-hand side of (5.2) $b\,(\omega) = b_t^{\circ}\,(\omega) \equiv b^{\circ}\,(t,\ \omega)$, and so on.

2°. For the incomplete principle of minimum to be valid for the group of parameters $\{a,\ c\}$ it is sufficient that the assumptions made in Sect. 1 are satisfied. For the validity of that principle for a group of parameters containing parameter $\{b\}$ it is necessary that condition (4.2) is satisfied.

3°. If $D^i\,(t),\ i = a,\ b,\ c$, is assumed to be a set of vectors from D^i that are independent of the event ω, then (5.2) represents the necessary condition of optimality in the problem with determinate control.

6. Example. It can be shown that operator R_0 introduced in Sect. 3 has a bounded inverse operator R_0^{-1} and that, consequently, the equality $(R_0^{-1})^* = (R_0^*)^{-1}$ is satisfied. This means that functions χ° and ν° can be determined by solving the conjugate problem

$$
(R_0^{-1})^*\chi^{\circ} = T_0, \quad (R_0^{-1})^*\nu^{\circ} = P_0 \qquad (6.1)
$$

When condition (4.2) is satisfied it is possible to obtain an explicit formula for operator $(R_0^{-1})^*$ and thus prove that the solution of problem (6.1) satisfies mes$\times P$-almost everywhere the identity

$$
\begin{aligned}
& \chi_i^{\circ}(t,\omega) - \frac{\partial A_k}{\partial x_i}\,(t,\eta^{\circ}(t,\omega),a^{\circ}(t,\omega),\omega)\,M\,\{\theta_k^{\circ}(t,\omega)\mid F_t\} - \qquad (6.2) \\
& \quad \frac{\partial B_{kj}}{\partial x_i}\,(t,\eta^{\circ}(t,\omega),b^{\circ}(t,\omega),\omega)\,\varkappa_{kj}^{\circ}(t,\omega) \equiv \frac{\partial J}{\partial x_i}\,(t,\eta^{\circ}(t,\ \omega),u^{\circ}(t,\omega),\omega)
\end{aligned}
$$

where $M\{\ \xi \mid F\ \}$ represents the conditional mathematical expectation of the random quantity ξ relative to the σ-algebra F.

Lemma 3. If $\partial B_{ij}\,/\,\partial x_k \equiv 0$ and the random function $y^{\circ}\,(t,\ \omega) \equiv \{y_1^{\circ}\,(t,\ \omega),\ \ldots,\ y_n^{\circ}\,(t,\ \omega)\}$ represents the solution of the system of ordinary differential equations with random right-hand sides

$$\frac{dy_i^\circ}{dt} = -y_k^\circ(t)\frac{\partial A_k}{\partial x_i}(t) - \frac{\partial J}{\partial x_i}(t), \quad y_i^\circ(T) = 0 \tag{6.3}$$

then the progressively measurable modification of the process

$$\chi_i^0(t) = \frac{\partial J}{\partial x_i}(t) + \frac{\partial A_k}{\partial x_i}(t) M\{y_k^\circ(t) \mid F_t\}$$

satisfies system (6.2).

Let us consider the simplest example. Let $\eta(t) \equiv \{\eta_1(t), \eta_2(t)\}$ be a controllable stochastic process specified by the system

$$d\eta_1 = (a_1 + \eta_2)dt + b_1 dw_1 \quad \eta_1(0) = \eta_1^\circ$$
$$d\eta_2 = (a_2 + \eta_1)dt + b_2 dw_1 \quad \eta_2(0) = \eta_2^\circ$$

In the class of determinate controls we have to determine a control $\{a_1^\circ, a_2^\circ; b_1^\circ, b_2^\circ\}$ whose values in the interval $[-1, 1]$ would minimize the functional

$$J(u) = M\int_0^T [T_1^\circ(s)\eta_1(s) + T_2^\circ(s)\eta_2(s)]\, ds$$

For simplicity we assume that the random processes T_1° and T_2° are martingales, i.e. $GT_i^\circ = \alpha_i^\circ(s, \omega)$, $i = 1, 2$.

Using (6.3) we obtain

$$\chi_i^\circ(t) = T_i^\circ(t)\,\mathrm{ch}\,(T-t) + T_j^\circ(t)\,\mathrm{sh}\,(T-t)$$
$$\lambda_i^\circ(t, s) = \alpha_i^\circ(s)\,\mathrm{ch}\,(T-t) + \alpha_j^\circ(s)\,\mathrm{sh}\,(T-t), \quad i, j = 1, 2; \quad i \neq j$$

Thus it follows from (5.2) that the controls

$$a_i^\circ(t) = -\mathrm{sgn}\,\{MT_i^\circ\,\mathrm{sh}\,(T-t) + MT_j^\circ(\mathrm{ch}\,(T-t) - 1)\}$$
$$b_i^\circ(t) = -\mathrm{sgn}\,M\,\{\alpha_i^\circ(t)\,\mathrm{sh}\,(T-t) + \alpha_j^\circ(t)(\mathrm{ch}\,(T-t) - 1)\}$$
$$i, j = 1, 2; \quad i \neq j$$

may prove to be optimal. It can be shown that it is in fact so.

REFERENCES

1. Krylov, N. V., On the control of solutions of stochastic integral equation. The Theory of Probability and Its Applications. Vol. 17, No. 1, 1972.

2. Fleming, W. H., Optimal control of partially observable diffusions. SIAM J. Control, Vol. 6, No. 2, 1968.

3. Kushner, H. J. and Schweppe, F. C., A maximum principle for stochastic control systems. J. Math. Analysis and Appl., Vol. 8, No. 2, 1964.

4. Kushner, H. J., On the stochastic maximum principle: Fixed time of control. J. Math. Analysis and Appl., Vol. 11, Nos. 1-3, 1965.

5. Plotnikov, V.I., Necessary conditions of optimality for controllable systems of the general form. Dokl. Akad. Nauk SSSR, Vol. 199. No. 2, 1971.

6. Plotnikov, V. I., The necessary and sufficient conditions of optimality and the conditions of uniqueness of optimizing functions for controllable systems of the general form. Izv. Akad. Nauk SSSR. Ser. Matem., Vol. 36, No. 3, 1972.

7. Gikhman, I.I.and Skorokhod, A.V., Stochastic Differential Equations. Kiev, "Naukova Dumka", 1968.

8. Novikov, A.A., On moment inequalities for stochastic integrals. The Theory of Probability and Its Applications. Vol. 16, No. 3, 1971.

9. Gikhman, I.I.and Skorokhod, A.V., The Theory of Random Processes. Vol. 3, Moscow, "Nauka", 1975.

10. Vantsel, A.D., Course of the Theory of Random Processes. Moscow, "Nauka", 1975.

11. Liptser, R.Sh. and Shiriaev, A.N., Statistics of Random Processes. Moscow. "Nauka", 1974.

Translated by J.J. D.